

# SYS-611 Homework #3

Due Mar. 3 2021

Submit the following using the online submission system: 1) Cover sheet with name, date, and collaborators, 2) Written responses in PDF format, 3) All work (e.g. .xlsx or .py files).

## 3.1 Simulating $\pi$ Again [5 points]

In class, we performed Buffon's needle dropping experiment to simulate the value of  $\pi$ . This problem illustrates a second common method to simulate the value of  $\pi$ .

Imagine dropping grains of rice onto the area shown where the gray region defined by  $x^2 + y^2 \leq 1$  represents a quarter of a unit circle (with radius 1) inside a square with area 1. If evenly distributed, the probability of a rice grain landing within the gray region is proportional to its area,

$$P\{\text{rice in gray}\} = \frac{\text{area of gray quarter circle}}{\text{area of white square}} = \frac{\pi \cdot 1^2/4}{1 \cdot 1} = \frac{\pi}{4}$$

We can simulate this problem by assigning the elementary random variables for the  $(X, Y)$  location of a randomly-dropped grain of rice:

$$X \sim \text{uniform}(0, 1)$$

$$Y \sim \text{uniform}(0, 1)$$

and compute the derived random variable  $Z$  signaling if the rice falls inside the gray area:

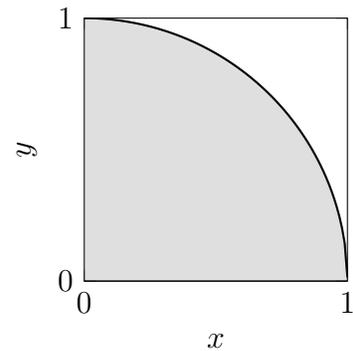
$$Z = \begin{cases} 1 & \text{if } \sqrt{X^2 + Y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where  $\mu_z = E[Z] = \frac{\pi}{4} \implies \pi = 4 \cdot \mu_z$ .

- 1 PT Identify or implement process generators for the two random variables  $X$  and  $Y$  and implement a derived state variable function  $Z$  based on the equation above.
- 2 PTS Perform a Monte Carlo simulation analysis and provide a 95% confidence interval for  $4 \cdot \mu_z$  using  $N = 1000$  samples (*Hint*: use samples  $4 \cdot Z$ , it should be close to  $\pi$ !).
- 2 PTS Create a second process generator using the antithetic derived state variable  $Z_a$ :

$$Z_a = \frac{Z_1 + Z_2}{2}, \quad Z_1 = \begin{cases} 1 & \text{if } \sqrt{X^2 + Y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad Z_2 = \begin{cases} 1 & \text{if } \sqrt{(1-X)^2 + (1-Y)^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Repeat the Monte Carlo simulation analysis and provide a 95% confidence interval for  $4 \cdot \mu_z$  using  $N = 1000$  samples (*Hint*: use samples  $4 \cdot Z_a$ , it should be close to  $\pi$ !).



## 3.2 Aircraft Range Estimation [9 points]

The Breguet range equation expresses aircraft range  $R$  (the distance it can fly without refueling) for steady, level flight as:

$$R = V \times \left(\frac{L}{D}\right) \times I_{sp} \times \ln\left(\frac{W_i}{W_f}\right)$$

where

$R$  : aircraft range [m]

$V$  : velocity of the aircraft [m/s]

$\frac{L}{D}$  : lift-to-drag ratio as a measure of aerodynamic efficiency [-]

$I_{sp}$  : specific impulse of the propulsion system as a measure of fuel efficiency [s]

$\frac{W_i}{W_f}$  : ratio of the initial (with fuel) to final (without fuel) weight of the aircraft [-]

During design of a new aircraft, some parameters are uncertain but constrained by lower- and upper-bound requirements. A systems engineer provides the following estimates:

$$V = 255 \text{ [m/s]}$$

$$\frac{L}{D} \sim \text{uniform}(a = 15, b = 18)$$

$$I_{sp} = 5950 \text{ [s]}$$

$$\frac{W_i}{W_f} \sim \text{ramp\_up}(a = 1.3, b = 1.5)$$

where  $\text{ramp\_up}(a, b)$  refers to a triangular distribution with lower bound  $a$  and mode/upper bound  $b$ .<sup>1</sup>

- (a) 2 PTS Identify or implement process generators for the random variables  $\frac{L}{D}$  and  $\frac{W_i}{W_f}$ .
- (b) 2 PTS Implement a function to compute the derived state variable  $R$  as a function of known and sampled variables.
- (c) 5 PTS Perform a Monte Carlo simulation analysis using  $N = 1000$  samples and report:
  - (i) Plot a histogram of the samples using appropriately-sized bins in kilometers [km]
  - (ii) Sample mean ( $\bar{R}$ ) in kilometers [km]
  - (iii) 25th percentile sample in kilometers [km] (*Hint*: find the 250th smallest sample)
  - (iv) Point estimate of probability of exceeding 9000 km range (*Hint*: find the number of samples with range above 9000 km divided by total number of samples.)

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<sup>1</sup>See Chapter 3 of Farr's course notes (p. 3-50) for help on ramp-up process generators. Note the ramp-up cumulative distribution function (CDF) is given by:

$$F(x) = \left(\frac{x-a}{b-a}\right)^2, \quad a \leq x \leq b$$

### 3.3 Sven's Sweets [11 points]

Sven's Sweets sells gourmet cookies. His wholesale orders for special occasions must be placed in 10-box cases. The "Shamrock Special" box costs Sven \$10 wholesale and sells for \$12 retail. Any boxes not yet sold by March 17 (St. Patrick's Day) are discounted by 50% (to \$6) to sell quickly and clear inventory space. Sales history data shows uniform demand between 40 and 90 boxes per year at full price (before St. Patrick's Day).

- (a) 2 PT What is the primary random variable of this problem? How is it distributed?  
(This is an elementary state variable for the model.)
- (b) 1 PT What is the "design" variable or decision which Sven must make?  
(This is also an elementary state variable for the model.)
- (c) 2 PTS What is the derived state variable which Sven seeks to maximize? Write it as a mathematical function in terms of the primary random variable in (a) and the design variable in (b). (*Hint:* define piece-wise to handle multiple conditions.)
- (d) 2 PTS Implement a function to compute the derived state variable in (c) using a selected value for (b) and a process generator to sample a value for (a). Use built-in process generators if desired.
- (e) 4 PTS Perform Monte Carlo simulation experiments for various values of (b) and make a recommendation to Sven supported by data and/or plots to explain your analysis.