Stevens Institute of Technology

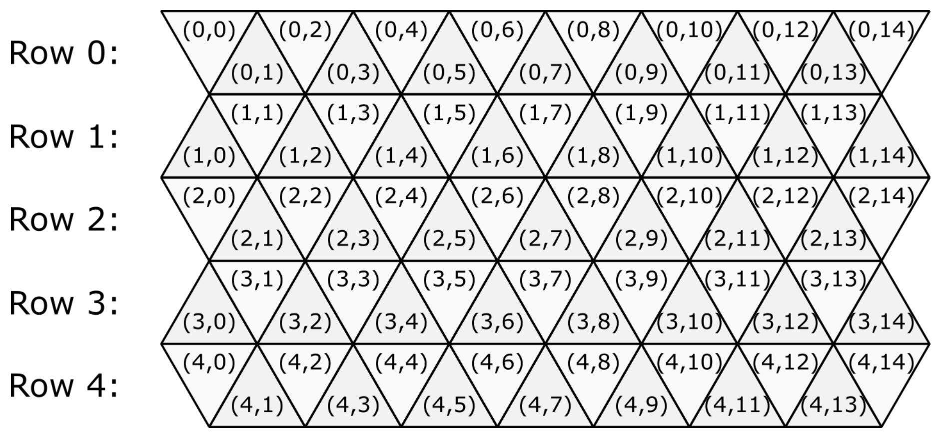
SYS-611 Homework #4

Due Mar. 24 2021

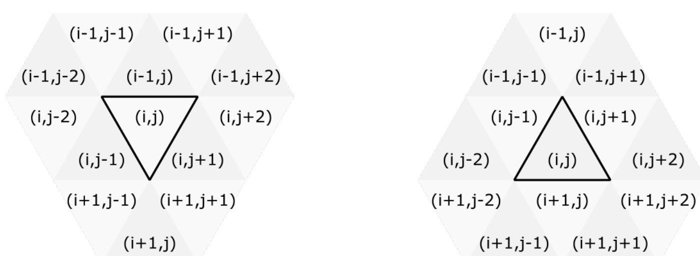
Submit the following using the online submission system: 1) Cover sheet with name, date, and collaborators, 2) Written responses in PDF format, 3) All work (e.g. .xlsx or .py files).

4.1 Triangle Man's Game of Life [8 points]

A variation on Conway's Game of Life takes place on a regular triangular lattice with R rows, C columns and $R \times C$ cells at locations (i, j) where $0 \leq i < R$ and $0 \leq j < C$ as shown below. An alive cell has state value $q_{i,j} = 1$; a dead cell has state value $q_{i,j} = 0$.



A neighbor is defined to be any cell with a shared apex or face (i.e. every interior triangle has 12 neighbors). There are two cases whether $(i + j)$ is even or odd shown below.



(i+j) even (i+j) odd

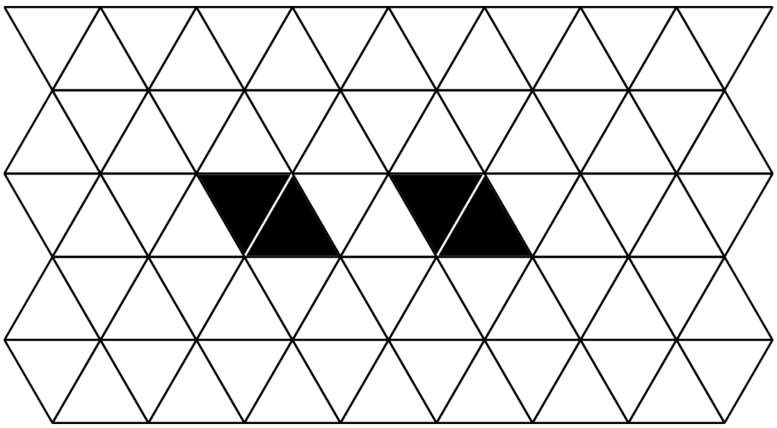
The rules for Triangle Man's Game of Life (version 2,3/3) are as follows:

1. An alive cell dies of over-crowding if four or more neighbors are alive.
2. An alive cell dies of loneliness if one or fewer neighbors are alive.

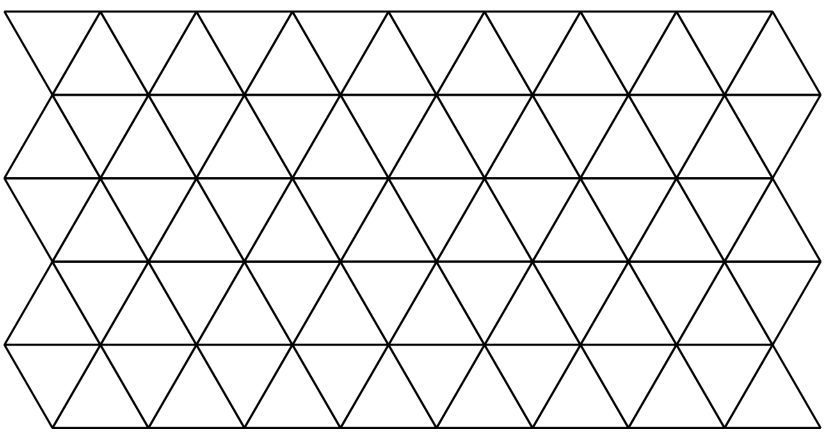
3. A dead cell becomes alive if exactly three neighbors are alive.

Don't worry about updating any cells touching the exterior world border.

(a) 2 PTS Calculate $N(i, j)$ for each cell in the initial state below (black = 1, white = 0). (*Hint: 4 cells with $N = 3$, and 12 with $N = 2$, and 12 with $N = 1$.*)



(b) 2 PTS Propagate the initial state above forward by 4 steps and report the result **after each step** using the blank template below (screen capture or download from Canvas). (*Hint: 4 alive cells at $t = 1$, 8 alive at $t = 2$, 6 alive at $t = 3$, 4 alive at $t = 4$.*)



(c) 2 PTS Define a mathematical function for the derived state variable $N(i, j)$ that counts the number of alive neighbors for cell $q_{i,j}$ in terms of other cell states, e.g., $q_{i-1,j}$, $q_{i,j+1}$. (*Hint: reference the neighboring state indices for $(i + j)$ even and $(i + j)$ odd cases.*)

$$N(i,j) = \begin{cases} \rule{7cm}{0.4pt} & \text{if } (i+j) \text{ even} \\ \rule{7cm}{0.4pt} & \text{if } (i+j) \text{ odd} \end{cases}$$

(d) 2 PTS Express the state update equation $\delta(q_{i,j})$ as a function of the elementary state variable $q_{i,j}$ and derived state variable $N(i, j)$.

$$\delta(q_{i,j}) = \begin{cases} 1 & \text{if } \rule{8cm}{0.4pt} \\ 0 & \text{if } \rule{8cm}{0.4pt} \end{cases}$$

4.2 Simulating a JK Flip-Flop [5 points]

A JK flip-flop is a digital circuit component that combines the functionality of a SR (set-reset), D (data), and T (toggle) flip-flops. It receives two inputs $x(t) = (j(t), k(t))$ and produces one (trivial) output $y(t) = \lambda(q) = q(t)$ with state transition function

$$\delta(q, x) = j(t) \cdot (1 - q(t)) + (1 - k(t)) \cdot q(t)$$

(a) 2 PTS Complete the output $\lambda(q)$ and state transition $\delta(q, x)$ values in the **transition/output table** below (*not a state trajectory table*) for a JK flip-flop.

$x(t)$		$q(t)$	$\lambda(q)$	$\delta(q, x)$
$j(t)$	$k(t)$			
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

(b) 3 PTS Simulate the **state and output trajectories** $q(t)$ and $y(t)$ for the following input trajectory $x(t) = (j(t), k(t))$ for $0 \leq t \leq 9$ with initial state $q(0) = 0$.

t:	0	1	2	3	4	5	6	7	8	9
$j(t)$:	0	1	0	1	1	0	0	1	0	0
$k(t)$:	1	0	1	1	1	0	1	0	1	0
$q(t)$:	0									
$y(t)$:										

4.3 Modeling COVID-19 [12 points]

A continuous time model represents the spread of COVID-19 across a population of fixed size N. The vector state variable

$$Q(t) = [S(t) \quad I(t) \quad R(t)]$$

composes the population in three categories: susceptible $S(t)$, infectious $I(t)$, and removed (either through recovery or death) $R(t)$. Note that $S(t) + I(t) + R(t) = N$ at all times and these quantities are treated as continuous variables.

The susceptible population *decreases* by a contact rate β times the susceptible population size $S(t)$ times the infectious population fraction $I(t)/N$.

$$\frac{dS}{dt} = -\beta S(t) \frac{I(t)}{N}$$

The infectious population *increases* by the contact rate β times the susceptible population size $S(t)$ times the infectious population fraction $I(t)/N$ and *decreases* by the removal rate γ times the infectious population size $I(t)$.

$$\frac{dI}{dt} = \beta S(t) \frac{I(t)}{N} - \gamma I(t)$$

The removed population *increases* by the removal rate γ times the infectious population size $I(t)$.

$$\frac{dR}{dt} = \gamma I(t)$$

Note that the inverse removal rate $1/\gamma$ is equivalent to the average infectious duration and the fraction $\beta/\gamma = R_0$ is called the *basic reproductive number*.

(a) 3 PTS Define the state transition function for each state variable using the Euler integration method. Write the functions in terms of the state variables $S(t)$, $I(t)$, and $R(t)$, model parameters N, β, and γ, and the integration time step Δt.

$$\delta(S, \frac{dS}{dt}, \Delta t) = \underline{\hspace{8cm}}$$
$$\delta(I, \frac{dI}{dt}, \Delta t) = \underline{\hspace{8cm}}$$
$$\delta(R, \frac{dR}{dt}, \Delta t) = \underline{\hspace{8cm}}$$

(b) 1 PT Assuming a basic reproductive number $R_0 = 3$ and an average infectious period of $1/\gamma = 10$ days, what is the contact rate β?

(c) 2 PTS Assume a total population (thousands) of $N = 1600$ with $I(0) = 1$ initially infected. Compute and plot the state variables $S(t)$, $I(t)$, and $R(t)$ for a duration of 180 days using transition functions in (a) and an integration time step of $\Delta t = 0.2$ day.

(d) 1 PT What is the peak number of infectious people and when does it occur?

(e) 1 PT Assuming 2% removed population mortality, how many deaths occur by day 180?

(f) Assume stay-at-home orders issued at $t = 0$ reduces the contact rate β to 0.2.

(i) 1 PT What is the new basic reproductive number R_0?

(ii) 1 PT Re-create the plot from (c) with the new values.

(iii) 1 PT What is the peak number of infectious people and when does it occur?

(iv) 1 PT Assuming 2% mortality, how many deaths occur by day 180?