Review of Probability and Statistics

SYS 611: Systems Modeling and Simulation

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Agenda

1. Samples and Statistics
2. Discrete Random Variables
3. Continuous Random Variables
4. Confidence Intervals

Reference: S.M. Ross, "Elements of Probability," "Random Numbers," Ch. 2-3 in *Simulation*, 2012.

J.V. Farr, "Review of Probability and Statistics," Ch. 3 in Simulation of Complex Systems and Enterprises, Stevens Institute of Technology, 2007.

Samples and Statistics

Samples and Statistics

A **sample** is an observation from a population

- Usually, cannot observe an entire population
- Each sample is subject to variation:
  - Aleatory variability: inherent uncertainty in process (randomness) – characterized by *random variables* (RVs)
  - Measurement error: calibration, environmental noise, instrument malfunction, human error, etc.

Statistics infer properties about a population

Descriptive Statistics

Descriptive statistics summarize observed samples and estimate properties about the population

- Sample mean $\bar{x}$ (estimate of population mean μ_x)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Sample variance s^2 (estimate of population variance σ_x^2)

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

- Sample standard deviation $s_x = \sqrt{s_x^2}$

Probability Basics

Probability quantifies chance of observing an event

- Observation of an event is either true (1) or false (0)
- What is the probability of a dice roll?
  - Outcome events: $D_1, D_2, D_3, D_4, D_5, D_6, D_{odd}, D_{even}$
  - Event probability: $P(D_6) = ?$ $P(D_{odd}) = ?$
- What is the probability of tomorrow's weather?
  - Outcome events: $W_{clear}, W_{cloudy}, W_{rain}, W_{snow}$
  - Event probability: $P(W_{rain}) = ?$ $P(W_{snow}) = ?$

Discrete Random Variables

Random Variables

Random variables assign events to numbers

- **Discrete random variables** assign events to *countable* numbers (e.g., integers)
  - Elementary events
  - Mutually exclusive and collectively exhaustive
- What is the *probability* of a dice roll?
  - Outcome event: X: random variable (value of roll)

$P\{X = 1\} = 1/6$ $P\{X = 2\} = 1/6$ $P\{X = 3\} = 1/6$

$P\{X = 4\} = 1/6$ $P\{X = 5\} = 1/6$ $P\{X = 6\} = 1/6$

$P\{X = 0\} = 0$ $P\{X = 7\} = 0$ $P\{X = -1\} = 0$

Example: Discrete R.V.

How many heads from flipping a fair coin three times?

$$X = \text{number of heads}$$

Sample space:

Event	$P(E)$	X
TTT	1/8	0
TTH	1/8	1
THT	1/8	1
THH	1/8	2

Event	$P(E)$	X
HTT	1/8	1
HTH	1/8	2
HHT	1/8	2
HHH	1/8	3

Distribution:

x	$P\{X = x\}$
0	1/8
1	3/8
2	3/8
3	1/8

Probability Mass Functions

Probability Mass Function (PMF) maps *discrete* random variables to probability *masses*

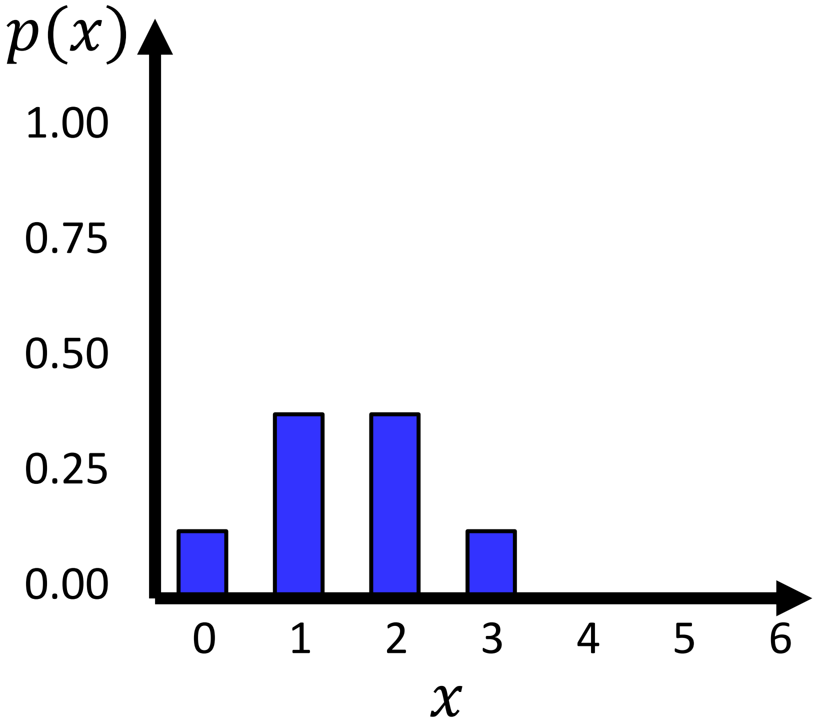
- Functional notation:
  - $P\{X = x\} = p(x)$
  - X: random variable (number of heads in 3 flips)
  - x: event outcome variable
- Note: $\sum_{x=-\infty}^{\infty} p(x) = 1$

$\boldsymbol{x}$	$\boldsymbol{p(x)}$
-1	0
0	1/8
1	3/8
2	3/8
3	1/8
4	0

PMF Plots

- PMF plots are *analytical* histograms
  - Replace frequency with probability mass

x	$p(x)$
-1	0
0	1/8
1	3/8
2	3/8
3	1/8
4	0



Cumulative Distribution Funct.

Cumulative Distribution Function (CDF) maps random variable ranges to probabilities

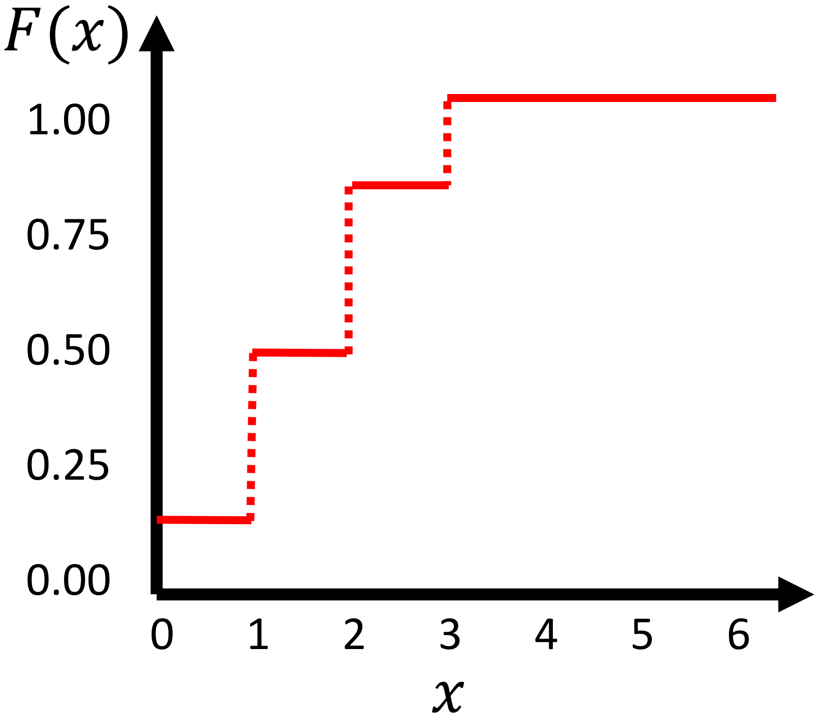
- Functional notation:
  - $P\{X \leq x\} = F(x)$
  - X: random variable (number of heads in 3 flips)
  - x: event outcome variable
- Note: $F(x) = \sum_{i=0}^{x} p(i)$

x	$p(x)$	$F(x)$
-1	0	0
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	8/8
4	0	8/8

CDF Plot

- CDF plots are *analytical* cumulative frequency plots
  - Replace cumulative freq. with cumulative prob.

x	$p(x)$	$F(x)$
-1	0	0
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	8/8
4	0	8/8



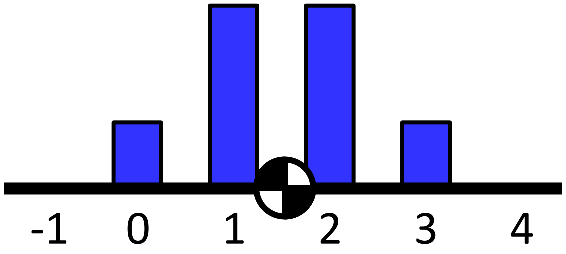
Mean or Expected Value

Expected value of a discrete distribution:

$$\mu_x = E[X] = \sum_{x=-\infty}^{\infty} x \cdot p(x)$$

Analogous to first moment (center of mass)

$$\sum_{x=0}^{3} x \cdot p(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \frac{3}{2}$$



Variance or Standard Deviation

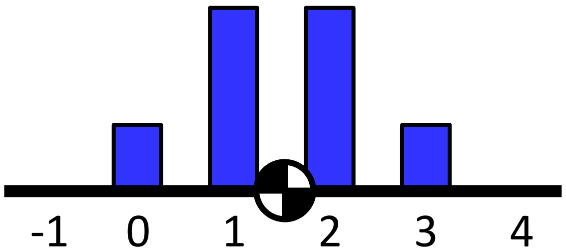
Variance of a discrete distribution:

$$\sigma_x^2 = Var(X) = E[(X - E[X])^2] = \sum_{x=-\infty}^{\infty} (x-\mu)^2 \cdot p(x)$$

$$= E[X^2] - (E[X])^2 = \sum_{x=-\infty}^{\infty} x^2 \cdot p(x) - \mu_x^2$$

Analogous to discrete form of moment of inertia

$$\sum_{x=0}^{3} (x-\mu)^2 \cdot p(x) = \left(0 - \frac{3}{2}\right)^2 \cdot \frac{1}{8} + \left(1 - \frac{3}{2}\right)^2 \cdot \frac{3}{8} + \left(2 - \frac{3}{2}\right)^2 \cdot \frac{3}{8} + \left(3 - \frac{3}{2}\right)^2 \cdot \frac{1}{8} = \frac{9 \cdot 1 + 1 \cdot 3 + 1 \cdot 3 + 9 \cdot 1}{4 \cdot 8} = \frac{3}{4}$$



Common Discrete Distributions

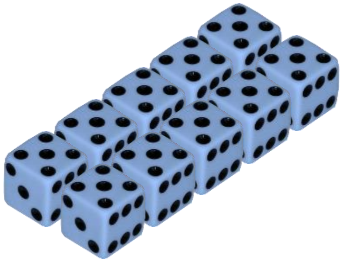
Name	Parameters	$p(x)$	$\mu_x = E[X]$	$\sigma_x^2 = Var(X)$
Uniform	Min $a \geq 0$, Max $b > a$, Num. $n > 0$	$\frac{1}{n}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Binomial	Trials $n > 0$, Prob. $p > 0$	$\binom{n}{x} p^x (1-p)^{n-x}$	$n \cdot p$	$n \cdot p \cdot (1-p)$
Geometric	Prob. $p > 0$	$p(1-p)^{x-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Rate $\lambda > 0$	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ
Hyper-geometric	Trials $n > 0$, Pop. $N > 0$, Succ. $0 < A < N$	$\frac{\binom{A}{x} \cdot \binom{N-A}{n-x}}{\binom{N}{n}}$	$n \cdot \frac{A}{N}$	$n \cdot \frac{A}{N} \cdot \frac{N-A}{N} \cdot \frac{N-n}{N-1}$
Negative Binomial	Successes $n > 0$, Prob. $p > 0$	$\binom{x+n-1}{n-1} p^n (1-p)^x$	$\frac{n \cdot (1-p)}{p}$	$\frac{n \cdot (1-p)}{p^2}$

Dice Roller Experiment

- The team has 10 dice and hits on a {3,4,5,6}.
- What is the probability of exactly $Y = y$ hits?
  - Y: # hits in 10 dice
  - $p(y) = P\{Y = y\}$
- Virtual dice: **random.org/dice**
- Post results: **pollev.com/pgrogan**

Blue Team:

- Small fighting force
- 4x effective weapons



- Roll 3|4|5|6 to hit target

Exercise: Dice Roller

- The team has 10 dice and hits on {3, 4, 5, 6}.
- What is the PMF/CDF for the number of hits?

$$Y\text{: number of hits from 10 dice,} \qquad P\{Y = y\} = p(y)$$

$$X_i\text{: } i\text{th die scores a hit,} \qquad P(X_i) = \frac{4}{6} = \frac{2}{3}$$

$$p(10) = \binom{10}{10} \cdot P(X_1) \cdot \ldots \cdot P(X_{10}) = \left(\frac{4}{6}\right)^{10}$$

$$\Rightarrow p(y) = \binom{10}{y}\left(\frac{2}{3}\right)^{y}\left(1 - \frac{2}{3}\right)^{10-y}$$

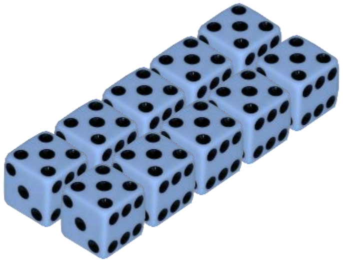
$$p(y) = \binom{n}{y} p^y (1-p)^{n-y}, \qquad Y \sim \text{binomial}\left(n = 10, p = \frac{2}{3}\right)$$

Critical Hit Experiment

- The team has 10 dice and hits on a {3,4,5,6}.
- Y: # hits in 10 dice
- $Y \geq 9$ is a "critical hit"
- What is the probability of observing $K = k$ "critical hits" in 10 seconds?
  - K: # critical hits in 10 sec
  - $p(k) = P\{K = k\}$

Blue Team:

- Small fighting force
- 4x effective weapons



- Roll 3|4|5|6 to hit target

Exercise: Critical Hit

- Number of "critical hits" in 10 seconds is a Poisson process:

$$K \sim \text{poisson}(\lambda), \qquad p(k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

- Probability of "critical hit" is:

$$P(\text{Crit}) = \left(\frac{4}{6}\right)^{10} + 10\left(\frac{4}{6}\right)^{9}\left(\frac{2}{6}\right) = \frac{1024 + 5120}{59049} = \frac{2048}{19683} = 0.104$$

- N students rolling dice, each roll takes T seconds
- Expect long-term average "critical hit" rate to be

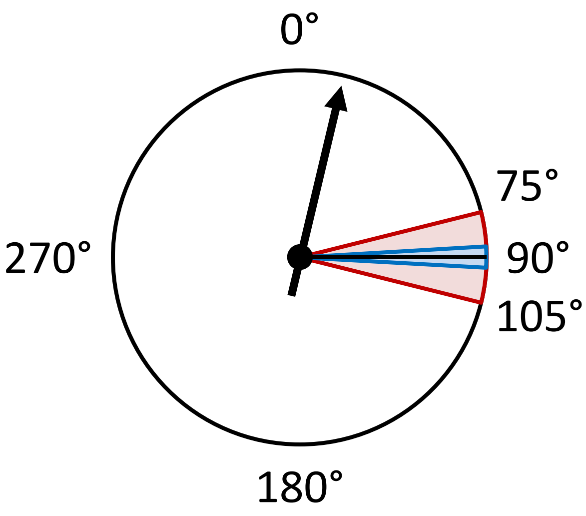
$$\lambda = P(\text{Crit}) \cdot \frac{N}{T} \text{ per second} = \frac{P(\text{Crit}) \cdot N}{(T/10)} \text{ per 10 seconds}$$

Continuous Random Variables

Continuous Random Variables

Random variables assign events to numbers

- Continuous random variables assign events to *uncountable* numbers (e.g., floating-point)
- What is the probability of a spinner stopping at a certain angle?
  - Outcome event: X: random variable (angle in degrees)
  - Event probability:
    - $P\{75.000 \leq X \leq 105.000\} = \frac{30}{360} = \frac{1}{12}$
    - $P\{85.000 \leq X \leq 95.000\} = \frac{10}{360} = \frac{1}{36}$
    - $P\{X = 90.000\} = 0$



Probability Density Functions

Probability Density Function (PDF) maps *continuous* random variables to probability *densities*

- Functional notation:
  - $f(x) \approx P\{x \le X \le x + \Delta x\}/\Delta x$
  - X: random variable
  - x: event outcome
  - Δx: small range of outcome (volume)
- Magnitude of $f(x)$ is selected such that

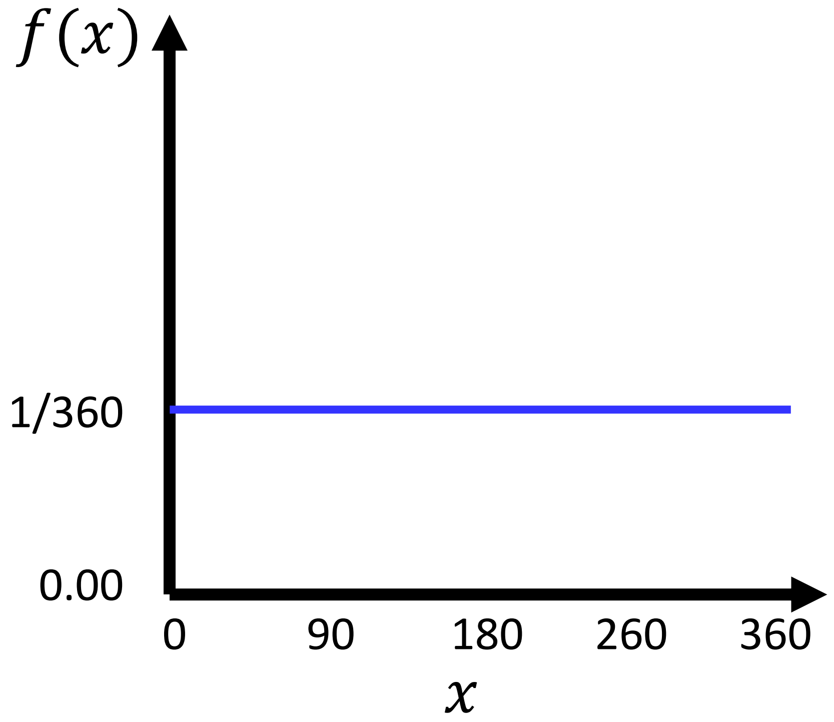
$$\int_{-\infty}^{\infty} f(x)dx = 1, \qquad \int_{0}^{360} f(x)dx = 1 \Rightarrow f(x) = 1/360$$

x	$f(x)$
0.0	1/360
10.0	1/360
23.4	1/360
89.9	1/360
90.0	1/360
180.0	1/360
359.9	1/360
360.1	0

PDF Plots

- PDF plots are like histograms, but use line charts instead of bar charts

x	$f(x)$
0.0	1/360
10.0	1/360
23.4	1/360
89.9	1/360
90.0	1/360
180.0	1/360
359.9	1/360
360.1	0



Cumulative Distribution Funct.

Cumulative Distribution Function (CDF) maps random variable ranges to probabilities

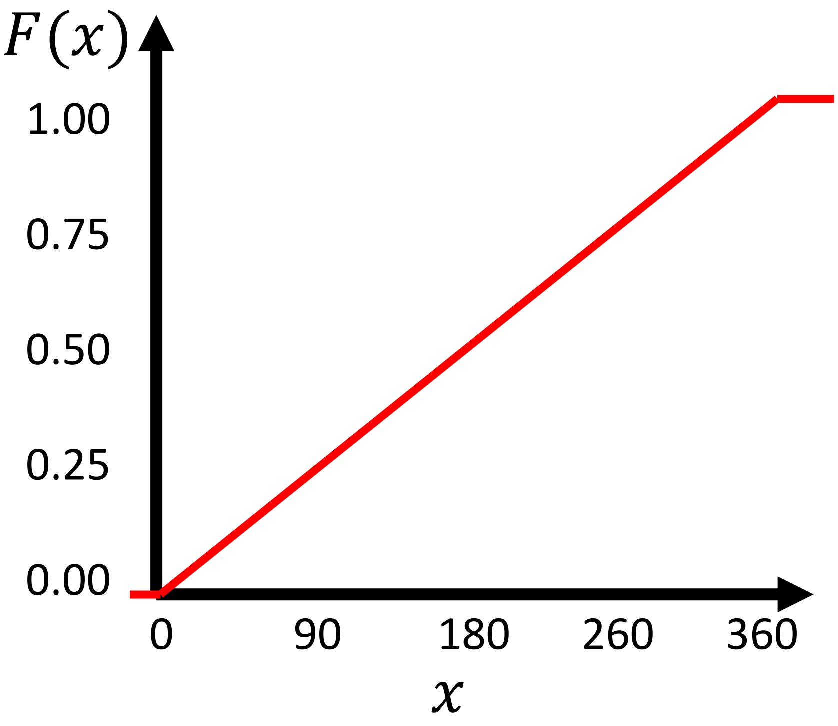
- Functional notation:
  - $P\{X \leq x\} = F(x)$
  - X: random variable
  - x: event outcome
- $F(x) = \int_{-\infty}^{x} f(i) di$

x	$f(x)$	$F(x)$
0.0	1/360	0.0/360
10.0	1/360	10.0/360
23.4	1/360	23.4/360
89.9	1/360	89.9/360
90.0	1/360	90.0/360
180.0	1/360	180/360
359.9	1/360	359.9/360
360.1	0	1

CDF Plot

- CDF plots are like cumulative frequency plots
  - Replace cumulative freq. with cumulative prob.

x	$f(x)$	$F(x)$
0.0	1/360	0.0/360
10.0	1/360	10.0/360
23.4	1/360	23.4/360
89.9	1/360	89.9/360
90.0	1/360	90.0/360
180.0	1/360	180/360
359.9	1/360	359.9/360
360.1	0	1



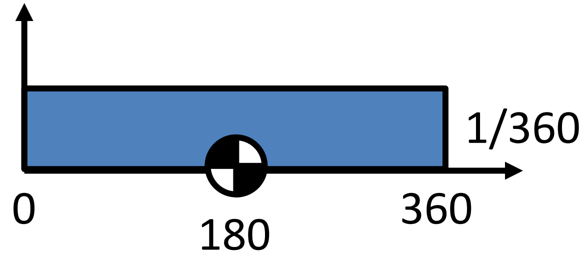
Mean or Expected Value

Expected value of a continuous distribution:

$$\mu_x = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

Analogous to the first moment (center of mass)

$$\int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{360} x \cdot \frac{1}{360} \cdot dx = \frac{1}{360} \cdot \frac{x^2}{2}\bigg|_0^{360} = \frac{360^2 - 0^2}{360 \cdot 2} = 180$$



Variance or Standard Deviation

Variance of a continuous distribution:

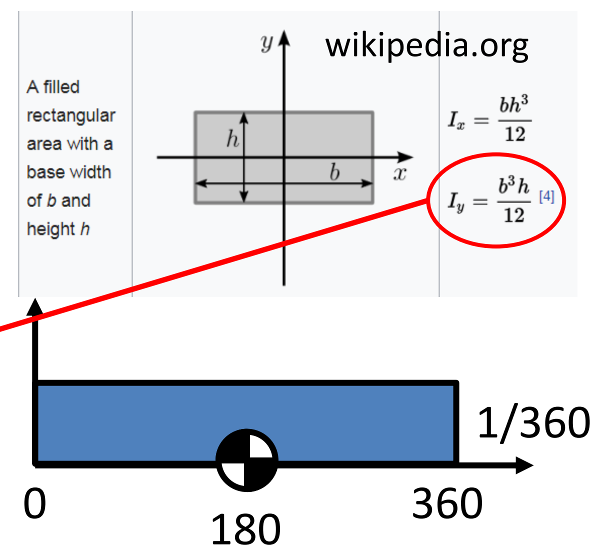
$$\sigma_x^2 = Var(x) = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) \cdot dx$$

$$= E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx - \mu_x^2$$

Analogous to moment of inertia:

$$\int_0^{360} x^2 \cdot \frac{1}{360} \cdot dx - 180^2 = \frac{1}{360} \cdot \frac{x^3}{3}\Big|_0^{360} - 180^2$$

$$= \frac{360^2}{3} - 180^2 = 10800 = \frac{1}{12} \cdot 360^3 \cdot \frac{1}{360}$$



Common Continuous Distributions

Name	Parameters	$f(x)$	$\mu_x = E[X]$	$\sigma_x^2 = Var(X)$
Uniform	Min a, Max $b > a$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Ramp Up	Min a, Max $b > a$	$\frac{2(x-a)}{(b-a)^2}$	$\frac{a+2b}{3}$	$\frac{a^2-2ab+b^2}{18}$
Triangular	Min a, Mode $c > a$, Max $b > c$	$\begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & a \leq x < c \\ \frac{2(b-x)}{(b-a)(b-c)} & c < x < b \end{cases}$	$\frac{a+b+c}{3}$	$\frac{a^2+b^2+c^2-ab-ac-bc}{18}$
Normal	Mean μ, Std. Dev. σ	$\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Standard Normal	(none)	$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$	0	1

Common Continuous Distributions

Name	Parameters	$f(x)$	$\mu_x = E[X]$	$\sigma_x^2 = Var(X)$
Chi-squared	Degrees of freedom k	$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$	k	$2k$
Exponential	Rate λ	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang	Rate λ, Shape k	$\frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}$	$\frac{k}{\lambda}$	$\frac{k}{\lambda^2}$
Gamma	Rate λ, Shape k	$\frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}$	$\frac{k}{\lambda}$	$\frac{k}{\lambda^2}$
Weibull	Rate λ, Shape α	$\alpha \lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}$	$\frac{1}{\lambda}\Gamma\left(1+\frac{1}{\alpha}\right)$	$\frac{1}{\lambda^2}\left[\Gamma\left(1+\frac{2}{\alpha}\right)+\left(\Gamma\left(1+\frac{1}{\alpha}\right)\right)^2\right]$

Exercise: Critical Hit

- Previously studied the number of events per time:
  - K: number of "critical hits" in 10 seconds
  - λ: average rate of "critical hit" events (per 10 seconds)

$$K \sim \text{poisson}(\lambda), \qquad p(k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

- Now study the time between adjacent events:
  - T: time between "critical hit" events
  - λ: average rate (per 10 seconds) of "critical hit" events

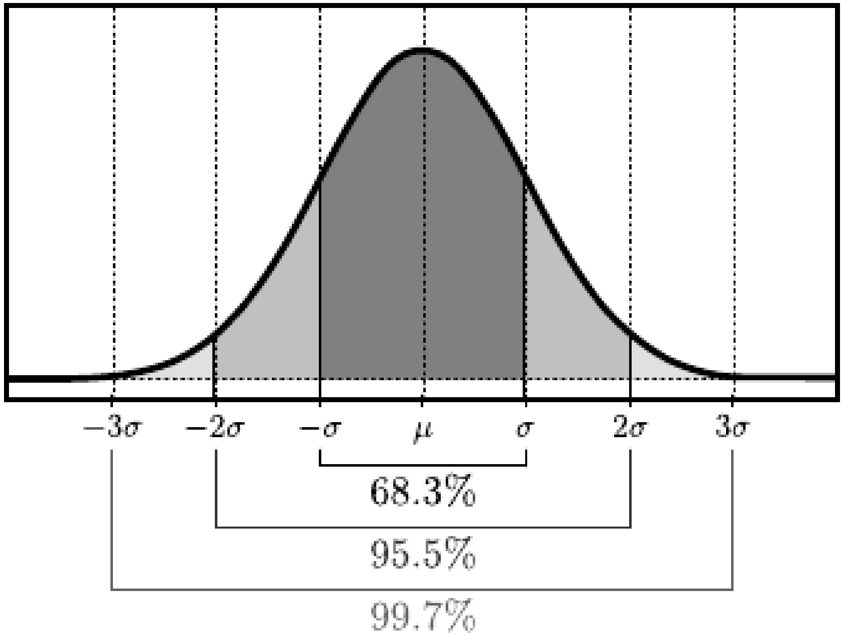
$$T \sim \text{exponential}(\lambda), \qquad f(t) = \lambda \cdot e^{-\lambda \cdot t}, \qquad F(t) = 1 - e^{-\lambda \cdot \mathrm{t}}$$

Confidence Intervals

Normal Distribution

Normal distribution (Gaussian) describes some natural random variables (e.g., human height)

- Mathematically describes the summation of many random variables
  - Used in SYS 611 to model distribution of sample means
  - Collect 1 sample mean but infer population statistics



Central Limit Theorem

Central Limit Theorem (CLT) states the sample mean of independent samples approaches a normal distribution regardless of the source population distribution.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables, each with finite mean μ_x and variance σ_x^2. Then the distribution of

$$\frac{(X_1 + X_2 + \cdots + X_n)}{n} = \bar{X}$$

approaches a **normal distribution** as $n \to \infty$.

$$\bar{X} \sim \text{normal}\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right)$$

Dice Roller Sample Mean

$$X \sim \text{binomial}(n, p)$$

$$\mu_x = n \cdot p$$

$$\sigma_x = \sqrt{n \cdot p \cdot (1 - p)}$$

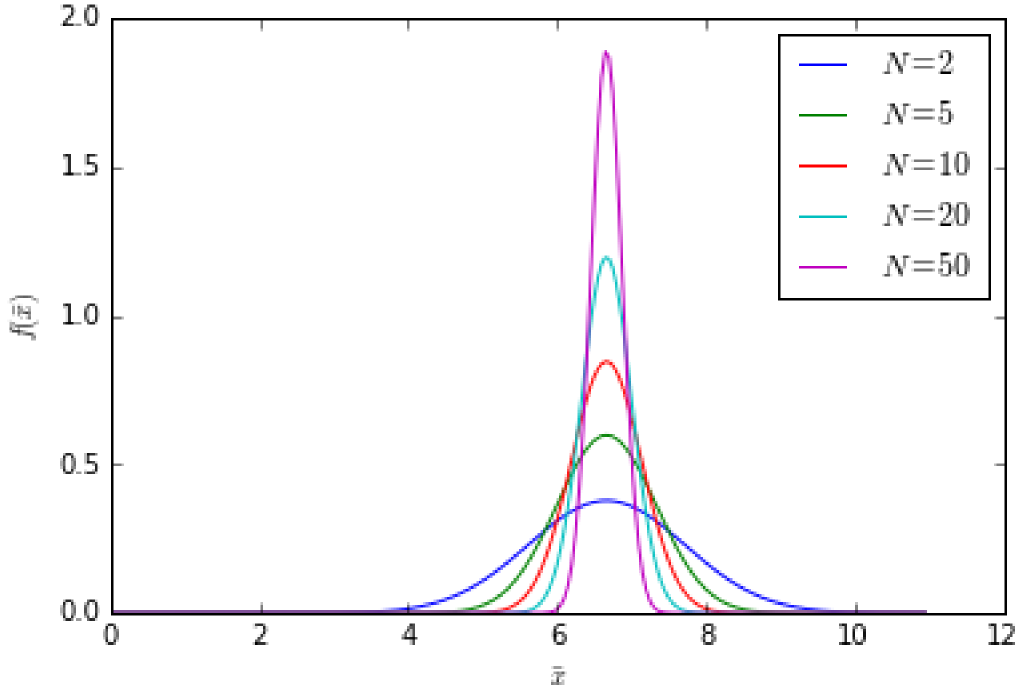
Sample mean from N samples:

$$\bar{X} \sim \text{normal}(\mu_{\bar{x}}, \sigma_{\bar{x}})$$

$$\mu_{\bar{x}} = \mu_x = n \cdot p = 10 \cdot \frac{2}{3} = 6.67$$

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \frac{\sqrt{(n \cdot p \cdot (1 - p)}}{\sqrt{N}}$$

$$= \frac{\sqrt{10 \cdot 2/3 \cdot 1/3}}{\sqrt{N}} = \frac{1.49}{\sqrt{N}}$$



Confidence Intervals

Confidence intervals apply the CLT to infer the population mean based on a number of samples:

- $(1 - \alpha) * 100\%$ confidence interval:

$$\bar{x} \pm z_{1-\alpha/2} \frac{\sigma_x}{\sqrt{N}}$$

- Critical z-score: $z_{1-\alpha/2}$

$$z_{0.975} = \text{normal}^{-1}(0.975,0,1) = 1.96$$

- Standard error of mean (SEM): $\frac{\sigma_x}{\sqrt{N}} \approx \frac{s_x}{\sqrt{N}}$

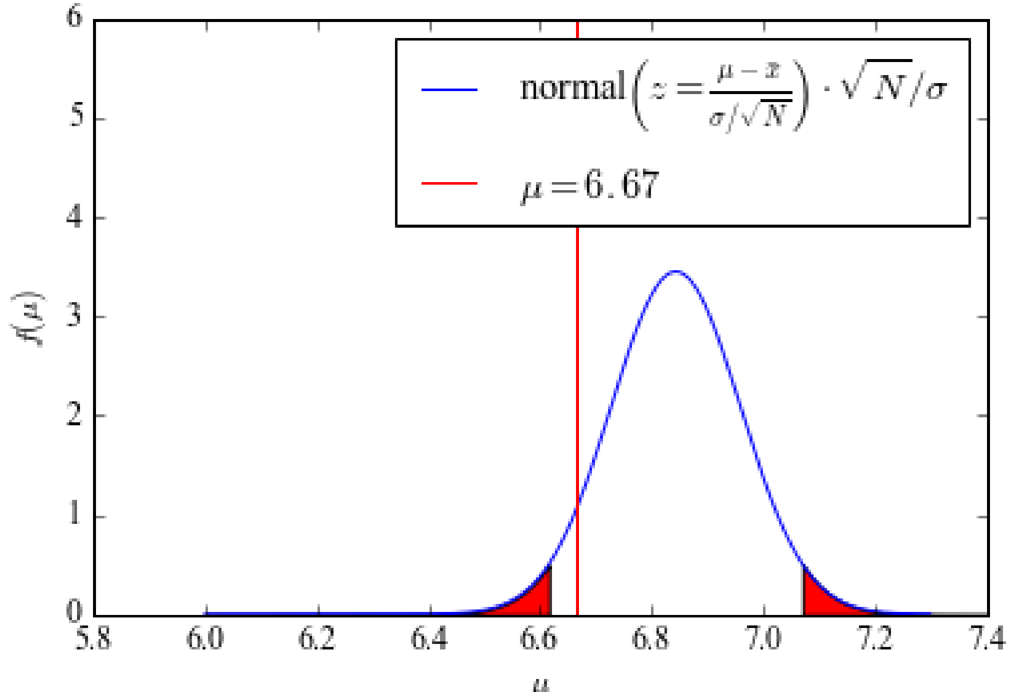
Estimating Population Mean

- Consider 166 samples with following statistics:
- $\bar{x} = 6.84,\ s_x = 1.49$
- $\mu_x \in \bar{x} \pm z_{1-\alpha/2} \frac{s_x}{\sqrt{N}}$

$$= 6.84 \pm 1.96 \cdot \frac{1.49}{\sqrt{166}}$$

$$= [6.62, 7.07]$$

- Population mean will fall in the range $[6.62, 7.07]$ 95% of the time



$$\bar{x} - z_{0.975} \frac{s_x}{\sqrt{N}} = 6.62$$

$$\bar{x} + z_{0.975} \frac{s_x}{\sqrt{N}} = 7.07$$